

Dynamical dimensional reduction in toy models of 4D causal quantum gravity

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In recent years several approaches to quantum gravity have found evidence for a scale dependent spectral dimension of space-time varying from four at large scales to two at small scales of order of the Planck length. The first evidence came from numerical results of four-dimensional causal dynamical triangulations (CDT) [Ambjørn et al., Phys. Rev. Lett. **95** (2005) 171]. Since then little progress has been made in analytically understanding the numerical results coming from the CDT approach and showing that they remain valid when taking the continuum limit. In this letter we propose a new toy model of “radially reduced” four-dimensional CDT in which we can take the continuum limit analytically and obtain a scale dependent spectral dimension varying from four to two with scale. Furthermore, the functional behaviour of the spectral dimension is exactly of the form which was conjectured on the basis of the numerical results.

Introduction – The quest to reconcile classical general relativity and quantum mechanics has a long history. One of the main difficulties is the fact that gravity is perturbatively non-renormalizable in four dimensions as was shown by ’t Hooft and Veltman in the seventies [1]. However, defining quantum gravity non-perturbatively, there is evidence from different approaches that there might still be a non-trivial ultraviolet fixed-point as suggested by Weinberg [2]. The causal dynamical triangulation (CDT) approach to quantum gravity (see [3] and [4] for a review) recently gave a surprising answer to what dynamical mechanism might regulate the theory at short distances in such a scenario. In particular, numerical simulations [5] show evidence for the dynamical reduction of the so-called spectral dimension from four at large scales to two at small scales of order of the Planck length (see also [6] for a discussion in a three-dimensional setting). More recently similar results have also been observed in other approaches to quantum gravity, most notably the exact renormalization group approach [7] and Hořava-Lifshitz gravity [8]. This also points towards several possible relations between those approaches [9].

Previous numerical insights from four-dimensional CDT – The idea behind the spectral dimension as a scale dependent measure of dimensionality is the following. Consider a diffusion process on a fixed space-time geometry. The diffusion kernel K_g is determined by the diffusion equation

$$\frac{\partial}{\partial \sigma} K_g(y, y_0; \sigma) = \Delta_g K_g(y, y_0; \sigma), \quad (1)$$

where g denotes the space-time metric, y_0 the starting point of the diffusion and y the position of the diffusion process after time σ . We can then define the return probability (density) by choosing starting and end point equal and integrating over all positions

$$P_g(\sigma) = \frac{1}{V_g} \int d^d y \sqrt{\det g_{ab}(y)} K_g(y, y; \sigma) \quad (2)$$

with $V_g = \int d^d y \sqrt{\det g_{ab}(y)}$. If we consider diffusion on quantum space-time we would have to take the ensemble average, formally defined using the gravitational path

integral

$$\langle P(\sigma) \rangle_Z = \frac{1}{Z} \int \mathcal{D}[g_{ab}] e^{-S_E(g_{ab})} P_g(\sigma), \quad (3)$$

where $Z = \int \mathcal{D}[g_{ab}] e^{-S_E(g_{ab})}$ is the partition function and $S_E(g_{ab})$ the Euclidean Einstein Hilbert action.

Formally, the scale dependent spectral dimension is then defined as [5]

$$D_s(\sigma) = -2 \frac{d \log \langle P(\sigma) \rangle_Z}{d \log \sigma}, \quad (4)$$

where σ is the diffusion time and corresponds to the scale at which the diffusion process probes the quantum geometry.

In the CDT approach the gravitational partition function is defined as a sum over four-dimensional simplicial geometries with a strict time-sliced structure

$$Z = \sum_T \frac{1}{C_T} e^{-S_E(T)}, \quad (5)$$

where C_T is a combinatorial symmetry factor and $S_E(T)$ the Euclidean Einstein-Regge action of the triangulation T ; $S_E(T) = \lambda N_4(T) - \nu N_2(T)$. Here λ is the bare cosmological constant and ν is the bare inverse Newton’s constant. $N_4(T)$ denotes the number of four-simplices and $N_2(T)$ the number of two-simplices in T .

In theory, one would like to evaluate the partition function (5) and take the infinite volume limit in which λ is tuned towards its critical value and ν is expressed in terms of the inverse renormalized Newton’s constant $1/G$. While an analytical evaluation of (3) in four-dimensional CDT, i.e. using the partition function (5), is out of reach, Monte Carlo simulations of random walks (discrete diffusion) on four-dimensional CDT [5] yield a scale dependent spectral dimension given by

$$D_s(\sigma) = 4.02 - \frac{119}{54 + \sigma} = \begin{cases} 1.80 \pm 0.25, & \sigma \rightarrow 0 \\ 4.02 \pm 0.1, & \sigma \rightarrow \infty. \end{cases} \quad (6)$$

A possible problem with (6) is that the simulations are inevitably affected by finite size effects and the dimensional reduction observed might simply be an artefact of

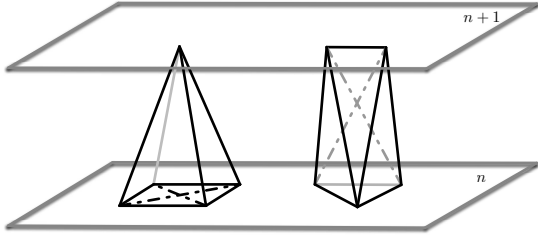


Figure 1. The two building blocks of four-dimensional CDT; The (4,1) and (3,2) simplex are on the left and right respectively. Dotted lines correspond to 3 simplices.

the discreteness scale. However, if one assumes that this expression can be extrapolated to continuum physics, one would conjecture the following expression for the return probability (density) (3) for four-dimensional CDT in the *infinite volume limit* [5]¹

$$\langle P(\sigma) \rangle_Z \sim \frac{1}{\sigma^2} \frac{1}{1 + \text{const. } G/\sigma}. \quad (7)$$

In this letter we introduce a new toy model of “radially reduced” four-dimensional CDT, a particular so-called *multigraph ensemble* in which the spectral dimension can be calculated analytically. At this point we would like to remind the reader of the difficulty caused by the two kinds of randomness – we are describing a *random* walk on a *random* geometry. The idea behind the “radial reduction” is that what essentially determines the properties of the random walk are the *volume growth* and the *resistance growth* (viewing the graph as an electrical network [10]). Hence, by omitting degrees of freedom which do not influence these quantities we get an effective description of the random walk on the original CDT ensemble.

Many of the more technical aspects of the present work, such as mathematically rigorous proofs, will be left for a forthcoming publication [11].

Multigraphs as toy models for causal quantum gravity – Consider four-dimensional rooted causal triangulations [12] of topology $I \times \Sigma^3$ which are made of (4,1) and (3,2)-simplices connecting vertices at distance n to vertices at distance $n+1$ from the root (Figure 1). Let us denote by $T(t)$ the ball of radius t around the root. This subset of the triangulation is characterised by its bulk variables $N_i(t) \equiv N_i(T(t))$, $i = 0, 1, 2, 3, 4$ which denote the number of i -simplices in this section of the triangulation. We can further distinguish these variables; for example, there are two different four-simplices so $N_4(t) = N_4^{(4,1)}(t) + N_4^{(3,2)}(t)$. Further, there are three different types of three-simplices and two different types

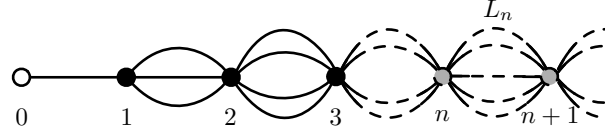


Figure 2. An example of a multigraph.

of triangles and links (i.e. space-like and time-like). These ten different bulk variables are related by seven topological constraints, leading to only three independent variables [12].

We can now define a multigraph from a given causal triangulation by the injective map in which we shrink all vertices at a fixed graph distance n from the root to a single point (see Figure 2). Let us denote the number of time-like edges connecting vertex n to vertex $n+1$ by L_n . The multigraph M is then completely characterised by the sequence $M = \{L_n, n = 0, 1, 2, \dots\}$. Furthermore, the partition function (5) induces a measure $\mu(\nu)$ for the $\{L_n, n = 0, 1, 2, \dots\}$ which defines the multigraph ensemble. In the infinite volume limit this measure $\mu(\nu)$ depends only on the bare Newton’s constant $1/\nu$ as an external parameter.

In previous work on two-dimensional CDT [13] it was observed that multigraph ensembles provide us with a “radial” approximation of the causal triangulation. In particular, it appears that studying random walks on these ensembles captures the full behaviour of the spectral dimension in two-dimensional models of CDT [13]. This suggests that they might also serve as a realistic toy model for studying the spectral dimension of higher-dimensional CDT. However, the work [13] on two-dimensional CDT does not describe a scale dependent spectral dimension. In a forthcoming article [11] we show that by introducing a length scale in the model one can actually obtain a spectral dimension varying from two at large scales to one at small scales. This extends a toy model of scale dependent spectral dimension using random combs first described in [14] and can be extended to four-dimensional CDT, as we describe here.

While an analytical derivation of the measure $\mu(\nu)$ of the multigraph ensemble related to four-dimensional CDT is still absent, we know for example from computer simulations [15] that $\langle N_4(t) \rangle_Z \simeq t^4$. From the topological relations mentioned above we find for the number of time-like links $N_1^{\text{TL}}(t)$ (up to boundary terms) that

$$N_1^{\text{TL}} = 2N_0(t) + \frac{1}{2}N_4^{(3,2)}(t) - 3\chi(T(t)) \quad (8)$$

and using $N_0(t) \leq N_4(t)/5$ and $N_4^{(3,2)}(t) \leq N_4(t)$ we get

$$\langle N_1^{\text{TL}}(t) \rangle_Z \leq \text{const.} \langle N_4(t) \rangle_Z \simeq t^4. \quad (9)$$

This suggests that for the measure $\mu(\nu)$ of the multi-

¹ Note that throughout this letter we write “ \sim ” for equality up to multiplicative logarithmic corrections, while “ \simeq ” denotes equality up to a multiplicative constant.

graph ensemble related to four-dimensional CDT

$$\langle B(t) \rangle_{\mu(\nu)} \equiv \left\langle \sum_{n=0}^t L_n \right\rangle_{\mu(\nu)} \leq t^4. \quad (10)$$

Motivated by an analytical model of a multigraph ensemble with scale dependent spectral dimension derived from two-dimensional CDT, as mentioned above [11], we are led to the following ansatz on the expectation of the number of links

$$\langle L_N \rangle_{\mu(\nu)} \simeq \nu N^{3-\epsilon} + N \quad (11)$$

where $\epsilon > 0$ can be taken arbitrarily small. The use of ϵ here is for purely technical reasons and for all practical purposes one can think of it as being zero. This leads to the volume expectation of the form

$$\underline{c} \langle L_N \rangle_{\mu(\nu)} \leq \langle B(N) \rangle_{\mu(\nu)} \leq \bar{c} \langle L_N \rangle_{\mu(\nu)} \quad (12)$$

where \underline{c}, \bar{c} are positive constants. Further, we assume that the resistance from a vertex at distance N to infinity, i.e. $R(N) \equiv \sum_{n=N}^{\infty} \frac{1}{L_n}$, and connectivity at height N , i.e. L_N , are bounded above by

$$R(N) \leq \frac{N}{\langle L_N \rangle_{\mu(\nu)}} \psi_+(\sqrt{\nu} N^{\frac{2-\epsilon}{2}}) \quad (13)$$

$$L_N \leq \langle L_N \rangle_{\mu(\nu)} \psi(\sqrt{\nu} N^{\frac{2-\epsilon}{2}}) \quad (14)$$

for $N > N_0$ for almost all graphs of the ensemble (such as those obtained from self-averaging in the computer simulations) where $\psi(x)$ and $\psi_+(x)$ are functions which diverge and vary slowly at $x = 0$ and $x = \infty$. For example in the two dimensional model we know that $\psi(x) \simeq \psi_+(x) \simeq |\log x|$ [13]. These rather technical assumptions (13) and (14) which allow only slowly varying fluctuations around the average are actually quite natural as one can see in the lower-dimensional study [11]. Note that the finite resistance $R(N)$ implies that this multigraph ensemble is non-recurrent, i.e. $d_s \geq 2$.

Spectral dimension and dimensional reduction – We propose the multigraph model with measure $\mu(\nu)$ defined by (11), (13) and (14) as a toy model for four-dimensional CDT. In this section we show that it has a scale dependent spectral dimension which is two at small scales and four at large scales.

Let $p_M(t)$ be the probability that a random walker on a fixed multigraph M returns to the root 0 after time t . The spectral dimension is defined through $p_M(t)$ via the relation $p_M(t) \sim t^{-d_s/2}$ at large time. More precisely, using the generating function for the return probabilities

$$Q_M(x) = \sum_{t=0}^{\infty} p_M(t) (1-x)^{t/2} \quad (15)$$

we define the spectral dimension using

$$Q_M(x) \sim x^{-1+d_s/2} \quad \text{as} \quad x \rightarrow 0 \quad (16)$$

in the case where the random walk is recurrent ($d_s < 2$) and $Q_M(x)$ diverges as $x \rightarrow 0$. If the random walk is

non-recurrent, $Q_M(x)$ is finite and we define the spectral dimension through the derivative of $Q_M(x)$ of lowest degree which is diverging via the relation

$$Q_M^{(k)}(x) \sim x^{-1-k+d_s/2} \quad \text{as} \quad x \rightarrow 0 \quad (17)$$

for $2k \leq d_s < 2(k+1)$. As we will see later for $\mu(\nu)$ the expectation value of $Q_M(x)$ is finite while its first derivative diverges.

Given a fixed multigraph M the probability for a random walker at n to step next to $n+1$ is given by $p_n(M) = L_n/(L_{n-1} + L_n)$ and the probability that the next step is to $n-1$ is $1 - p_n(M)$ (note that the probability to move from the root to vertex one is 1). Then we decompose the random walk into two pieces; a step from vertex n to $n+1$, then a random walk returning to $n+1$ and a final step from $n+1$ to n at time t . This decomposition relates $Q_{M_n}(x)$ and $Q_{M_{n+1}}(x)$ and the generating function satisfies the following recursion relation [13]

$$\eta_{M_n}(x) = \eta_{M_{n+1}}(x) + \frac{1}{L_n} - x L_n \eta_{M_n}(x) \eta_{M_{n+1}}(x) \quad (18)$$

where $\eta_{M_n}(x) \equiv Q_{M_n}(x)/L_n$ and M_n is the multigraph obtained from M by removing the first n vertices and all edges attached to them and relabelling the remaining multigraph. Recall that $Q_M(x) \equiv Q_{M_0}(x)$. Differentiating (18) and iterating we get

$$\begin{aligned} |\eta'_{M_0}(x)| &= |\eta'_{M_N}(x)| \cdot \prod_{n=0}^{N-1} \frac{1 - x L_n \eta_{M_n}(x)}{1 + x L_n \eta_{M_{n+1}}(x)} \\ &+ \sum_{n=0}^{N-1} \frac{L_n \eta_{M_n}(x) \eta_{M_{n+1}}(x)}{1 + x L_n \eta_{M_{n+1}}(x)} \cdot \prod_{k=0}^{n-1} \frac{1 - x L_k \eta_{M_k}(x)}{1 + x L_k \eta_{M_{k+1}}(x)} \end{aligned} \quad (19)$$

which is the starting point of our proofs. Starting from (19) and using (13)-(14) we find that

$$\begin{aligned} c \frac{1}{x^2 B(N^*)} &< |\eta'_{M_0}(x)| < \\ c' \left(\frac{\eta_{M_{N^*}}(0)}{x} + \sum_{n=0}^{N^*-1} L_n \eta_{M_n}(0) \eta_{M_{n+1}}(0) \right) \end{aligned} \quad (20)$$

where $N^* = \lceil b x^{-1/2} \rceil$ and c, c' and b are positive constants. The detailed derivation of (20) is more involved and will be presented elsewhere [11]. Noting that $\eta_{M_N}(0) = R(N)$ is the (finite) resistance, averaging over the ensemble $\mu(\nu)$ and using (11)-(14) and Jensen's inequality we have that

$$c_- \frac{1}{x^{\frac{5}{2}} \nu + x} < \langle |\eta'_{M_0}(x)| \rangle_{\mu(\nu)} < c_+ \frac{\psi_+^2 \left(\nu^{\frac{1}{2}} x^{-\frac{1}{2} + \frac{\epsilon}{4}} \right)}{x^{\frac{5}{2}} \nu + x}. \quad (21)$$

We obtain the scaling limit by taking the lattice spacing $a \rightarrow 0$ and following the prescription of [14] define

$$|\tilde{Q}'(\xi, G)| \equiv \lim_{a \rightarrow 0} \left(\frac{a}{G} \right) \langle |Q'_M(x = a\xi)| \rangle_{\mu(\nu)} \quad (22)$$

with $\nu = a^{1-\frac{\epsilon}{2}}/G$. Using (20) we now have that

$$c_- \frac{1}{\xi^{\frac{\epsilon}{2}} + G\xi} \leq |\tilde{Q}'(\xi, G)| \leq c_+ \frac{\psi_+^2(G^{-\frac{1}{2}}\xi^{-\frac{1}{2}+\frac{\epsilon}{4}})}{\xi^{\frac{\epsilon}{2}} + G\xi}. \quad (23)$$

Eq. (23) implies that $d_s = 2$ in the short walk limit (i.e. $\xi \rightarrow \infty$) and $d_s = 4 - \epsilon$ in the long walk limit (i.e. $\xi \rightarrow 0$) for ϵ arbitrarily small (see [11] for the detailed correspondence between walk length and ξ). At this point we should mention that it is a highly non-trivial fact that there exists a non-trivial limit (22). The first example of the existence of such a limit (in the recurrent case) was given in the context of random combs [14].

We can extract from (21) the average return probability as a function of large walk length. In particular, using a Tauberian theorem one gets (setting ϵ to zero in the expressions below to simplify the discussion)

$$\langle p_M(t) \rangle_{\mu(\nu)} \sim \frac{2}{t^2} \left(\frac{\nu + 1}{(1 - 1/t)^2} - 1 \right)^{-1} \quad (24)$$

as $t \rightarrow \infty$. Scaling $t(a) = \lfloor \sigma/a \rfloor$ and $\nu(a) = a/G$ as before one obtains the probability density of the continuous diffusion time σ through

$$\tilde{P}(\sigma) \equiv \lim_{a \rightarrow 0} \left(\frac{a}{G} \right)^{-1} \langle p_M(t) \rangle_{\mu(\nu)} \sim \frac{2G^2}{\sigma^2} \frac{1}{1 + 2G/\sigma}. \quad (25)$$

This is precisely expression (7) which was conjectured in [5] as the behaviour of the continuum return probability density for diffusion on four-dimensional CDT. It yields the scale dependent continuum spectral dimension

$$D_s(\sigma) = 4 \left(1 - \frac{1}{2 + \sigma/G} \right) \quad (26)$$

consistent with the numerical results.

Discussion – We present a multigraph ensemble which describes “radially reduced” four-dimensional CDT. It is shown that starting from (11), (13) and (14), which are all motivated from lower-dimensional studies [11], one can calculate the spectral dimension of the resulting model analytically (see also [11] for many of the technical details of the derivation). One observes a scale dependent spectral dimension varying from four at large scales to two at small scales. In fact, one of the non-trivial results of this letter is that one can actually perform the continuum limit exactly and obtain an expression for the continuum return probability density (25). This result is compatible with the numerical results from [5] and furthermore gives a derivation of the conjectured expression (7). It is quite surprising that the above assumptions suffice to derive this result. While (11) only describes the expectation value of time-like edges at distance N from the root it would be interesting to further study correlations of the L_n from the numerical data and possibly derive their exact distribution. In particular, such a study should shed further light on the mechanism of dynamical dimensional reduction in four-dimensional CDT and would be a first step towards an analytical solution of the full four-dimensional model.

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